Lecture 43

16.9 - The Divergence Theorem

The Divergence Theorem

Let E be a bounded solid region in $\mathbb{R}^3$ with boundary S, where S consists of finitely many piecewise smooth, closed, orientable surfaces, each of which oriented with normals pointing away from E. Let $\vec{F}$ be a vector field which is C^1 on an open region containing E. Then

$$\boxed{\iint_S \vec{F}\cdot d\vec{S} = \iiint_E (\operatorname{div}\vec{F})\,dV}$$

Ex: Compute the flux of $\vec{F} = \langle z, y, x\rangle$ across the sphere $x^2+y^2+z^2 = 7921$, where the sphere is oriented outward.

Sol: $\operatorname{div}\vec{F} = 0+1+0$. This sphere bounds the ball $B: x^2+y^2+z^2 \le 7921 = 89^2$. By the Divergence theorem

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_B (\operatorname{div}\vec{F})\,dV = \iiint_B 1\,dV = \operatorname{vol}(B) = \frac{4}{3}\pi(89)^3$$

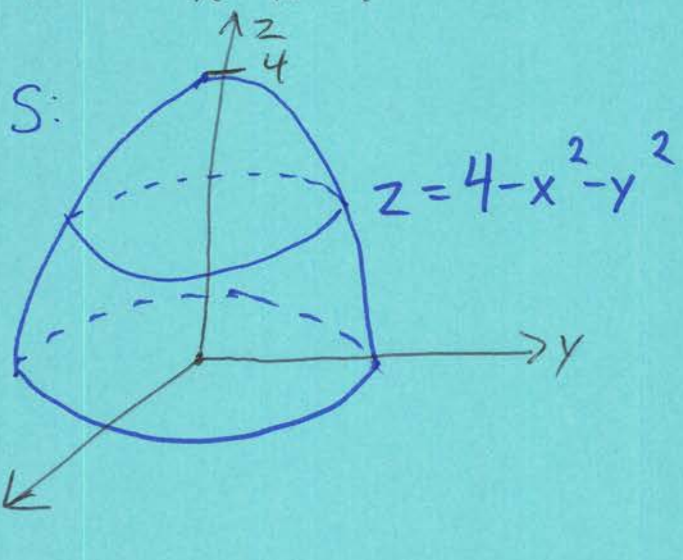
$$= \frac{4}{3}\pi(704969)$$

◇

<u>Ex</u>: Compute $\iint_S \vec{F}\cdot d\vec{S}$ where S is the boundary of the solid bounded by $z=4-x^2-y^2$ and the xy-plane, oriented positively, where $\vec{F}=\langle x^2, xy, z\rangle$.

<u>Sol</u>: $\text{div}\,\vec{F} = 2x+x+1 = 3x+1$

<u>Region</u>: S:



Let E be the solid bounded by S.

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E (\text{div}\,\vec{F})\,dV = \int_0^{2\pi}\int_0^2\int_0^{4-r^2}(3r\cos\theta+1)\,r\,dz\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^2\int_0^{4-r^2}(3r^2\cos\theta+r)\,dz\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^2 (3r^2\cos\theta+r)(4-r^2)\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^2 (12r^2\cos\theta+4r-3r^4\cos\theta-r^3)\,dr\,d\theta$$

$$=\int_0^{2\pi}\left[4r^3\cos\theta+2r^2-\frac{3}{5}r^5\cos\theta-\frac{1}{4}r^4\right]_0^2 d\theta$$

$$=\int_0^{2\pi}\left(32\cos\theta+8-\frac{96}{5}\cos\theta-4\right)d\theta=\int_0^{2\pi}\left(\frac{64}{5}\cos\theta+4\right)d\theta=8\pi$$

◇

The divergence theorem can also help in computing surface integrals over non-closed surfaces. This is done by "capping off" the surface, using the divergence theorem, then subtracting off the surface integral over the "cap". Formally:

If S_1 is a surface which isn't closed, but $S_1 \cup S_2$ is for some surface S_2, and $S_1 \cup S_2$ bounds a solid E and is oriented outward, then

$$\iint_{S_1} \vec{F} \cdot d\vec{S} + \iint_{S_2} \vec{F} \cdot d\vec{S} = \iint_{S_1 \cup S_2} \vec{F} \cdot d\vec{S} = \iiint_E (\operatorname{div} \vec{F})\, dV$$

$$\Rightarrow \iint_{S_1} \vec{F} \cdot d\vec{S} = \iiint_E (\operatorname{div} \vec{F})\, dV - \iint_{S_2} \vec{F} \cdot d\vec{S}$$

<u>Ex</u>: Let $\vec{F} = \langle x + z\arctan(y^3),\ z^3 \ln(x^2+1),\ 0 \rangle$. Compute the flux of $\vec{F}$ across the surface which is the part of the paraboloid $x^2+y^2+z=2$ above the plane $z=1$, oriented upward.

Sol: To cap this surface off, we fill in the hole at the bottom with the surface S_2 given by $x^2+y^2 \leq 1$, $z=1$. The correct orientation to give S_2 so that $S_1 \cup S_2$ has a continuous normal vector field is the downward orientation.

Then, $\iint_{S_1} \vec{F} \cdot d\vec{S} = \iiint_E (\operatorname{div} \vec{F}) \cdot dV - \iint_{S_2} \vec{F} \cdot d\vec{S}$

where E is the solid bounded by $S_1 \cup S_2$

$\operatorname{div} \vec{F} = 1+0+0=1$

So, $\iiint_E (\operatorname{div} \vec{F})\, dV = \int_0^{2\pi} \int_0^1 \int_1^{2-r^2} r\, dz\, dr\, d\theta = \int_0^{2\pi} \int_0^1 rz \Big|_1^{2-r^2} dr\, d\theta$

$= \int_0^{2\pi} \int_0^1 (2r-r^3-r)\, dr\, d\theta = \int_0^{2\pi} \int_0^1 (r-r^3)\, dr\, d\theta$

$= \int_0^{2\pi} \left(\frac{1}{2}r^2 - \frac{1}{4}r^4\right)\Big|_0^1 d\theta = \int_0^{2\pi} \frac{1}{4}\, d\theta = \frac{\pi}{2}$

Now, we compute $\iint_{S_2} \vec{F} \cdot d\vec{S}$. Parametrize S_2:

$\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, 1 \rangle$, $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$.

Find the downward normals for S_2:

$\vec{r}_r = \langle \cos\theta, \sin\theta, 0\rangle \quad \vec{r}_\theta = \langle -r\sin\theta, r\cos\theta, 0\rangle$

The downward normals are given by

$$\vec{r}_\theta \times \vec{r}_r = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & 0 \end{vmatrix} = \langle 0, 0, -r\rangle$$

$\vec{F}(\vec{r}(r,\theta)) = \langle r\cos\theta + \arctan(r^3\sin^3\theta), \ln(r^2\cos^2\theta+1), 0\rangle$

$\vec{F}(\vec{r}(r,\theta)) \cdot (\vec{r}_\theta \times \vec{r}_r) = 0$

So, $\iint_{S_2} \vec{F}\cdot d\vec{S} = 0$

Thus $$\iint_{S_1} \vec{F}\cdot d\vec{S} = \iiint_E (\operatorname{div}\vec{F})\,dV - \iint_{S_2} \vec{F}\cdot d\vec{S} = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$\diamond$